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The $(3, 3)$-colorability of planar graphs without 4-cycles and 5-cycles

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ABSTRACT

A graph G is called $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ such that the maximum degree of the induced subgraph $G[V_i]$ of G is at most d_i for $i \in \{1, \ldots, r\}$. Steinberg conjectured that every planar graph without 4/5-cycles is (0, 0, 0)-colorable. Unfortunately, the conjecture does not hold and it has been proved that every planar graph without 4/5-cycles is (1, 1, 0)-colorable. When only two colors are allowed to use, it is known that some planar graphs without 4/5-cycles are not $(1, k)$-colorable for any $k \geq 0$ and every planar graph without 4/5-cycles is $(3, 4)$-colorable or $(2, 6)$-colorable. In this paper, we reduce the gap for 2-coloring by proving that every planar graph without 4/5-cycles is $(3, 3)$-colorable.

1. Introduction

Graph coloring is one of the most extensively studied topics in graph theory and graph algorithms. In the traditional vertex coloring problem, we found an assignment of colors to the vertices of a graph such that no two adjacent vertices have the same color. A kind of generalization or relaxation of vertex coloring, called *improper coloring* or *defective coloring*, is to color the vertices of a graph allowing some adjacent vertices receiving the same color. Formally speaking, a graph G is called *improper $(d_1, \ldots, d_r)$-colorable*, or simply *$(d_1, \ldots, d_r)$-colorable*, if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ such that the maximum degree of the induced subgraph $G[V_i]$ of G is at most d_i for each $i \in \{1, \ldots, r\}$. In this paper, we are interested in the (improper) colorability of planar graphs.

The famous Four Color Theorem says that every planar graph is $(0, 0, 0, 0)$-colorable. Any graph without odd cycles is $(0, 0)$-colorable. Then under what conditions can the planar graph be colored by using three colors? Grötzsch Theorem shows that every triangle-free planar graph is (0, 0, 0)-colorable. Steinberg [18] conjectured in 1993 that every planar graph without 4-cycles and 5-cycles was (0, 0, 0)-colorable. In the following decades, there were many contributions toward to this research line and Steinberg's conjecture. Hill et al. [12] proved that every planar graph without 4-cycles and 5-cycles is (3, 0, 0)-colorable. Chen et al. [2] proved that every planar graph without 4-cycles and 5-cycles is (2, 0, 0)-colorable. Xu et al. [21] proved that every planar graph without 4-cycles and 5-cycles is (1, 1, 0)-colorable. Recently, Cohen-Addad et al. [6] disproved Steinberg's Conjecture by constructing a nice example. Whether every planar graph without 4-cycles and 5-cycles is (1, 0, 0)-colorable is unknown yet. Recently, people were also interested in improper colorability by using only two colors. Sittitrai and Nakprasit [17] first showed that not all planar graphs without 4-cycles and 5-cycles are $(1, k)$-colorable for each

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Table 1
Improper colorability results on planar graphs without 4/5-cycles.

Classes	Results	References
3-coloring	non-(0, 0, 0)	Cohen-Addad et al. [6]
	(3, 0, 0)	Hill et al. [12]
	(2, 0, 0)	Chen et al. [2]
	(1, 1 ,0)	Xu et al. [21]
2-coloring	non-(1, k) for all $k \geq 0$	Sittitrai and Nakprasit [17]
	(2, 9)	
	(3, 5)	
	(4, 4)	
	(2, 6)	Liu and Lv [14]
	(3, 4)	Cho et al. [3]
	(3, 3)	**This paper**

positive integer k. Then Liu and Lv [14] proved the (2, 6)-colorability and Cho et al. [3] proved the (3, 4)-colorability for planar graphs without 4-cycles and 5-cycles. In this paper, we further show that

Theorem 1. *Every planar graph without 4-cycles and 5-cycles is* (3, 3)*-colorable.*

The results on (improper) colorability of planar graphs without 4-cycles and 5-cycles are summarized in Table 1.

We also note some other (improper) colorability results of planar graphs. Cowen et al. [7] proved that every planar graph is (2, 2, 2)-colorable. Liu et al. [15] proved that every planar graph without 5-cycles and intersecting triangles is (1, 1, 0)-colorable. Hoskins et al. [13] proved that every planar graph without 4-cycles and close triangles is (2, 0, 0)-colorable. Dai et al. [9] proved that every planar graph without cycles of length 4 or 9 is (1, 1, 0)-colorable. For planar graphs with girth at least 6, Borodin and Kostochka [1] proved the (1, 4)-colorability and Havet and Seren [11] proved the (2, 2)-colorability. For planar graphs with girth at least 5, Choi et al. [4] proved the (1, 10)-colorability, Borodin and Kostochka [1] proved the (2, 6)-colorability, and Choi et al. [5] proved the (3, 4)-colorability.

For the computational complexity, it is NP-complete to check whether a graph is (a, b)-colorable for any nonnegative integers a and b except $a = b = 0$ [20] and NP-complete to check whether a planar graph is $(0, 0, 0)$-colorable [10]. Furthermore, Sittitrai and Nakprasit [17] showed that it is NP-complete to determine whether a planar graph without 4/5-cycles is (0, k)-colorable for every positive integer k. Montassier and Ochem [16] proved the NP-completeness of determining whether a planar graph with girth $g_{i,j}$ is (i, j)-colorable, where $g_{i,j}$ is the largest integer g such that there exists a planar graph with girth g that is not (i, j)-colorable.

2. Preliminaries

All graphs in this paper are finite and simple. A graph is planar if it has a drawing without crossings; such a drawing is a planar embedding of a planar graph. For a graph G, denote the vertex set, edge set and face set by $V(G)$, $E(G)$ and $F(G)$, respectively. The graph induced by a subset V' of vertices is denoted by $G[V']$. For a vertex subset V_1, we use $G - V_1$ to denote the graph $G[V \setminus V_1]$, and for an edge subset E_1, we let $G - E_1$ be the graph $G' = (V, E \setminus E_1)$. A set $\{v\}$ of a single element may be written as v directly. If a vertex or an edge is on the boundary of a face f, then we say f *contains* the vertex or edge. If a vertex is an endpoint of an edge, then we say that they are *incident with* each other. If a vertex (resp., an edge) is on the boundary of a face, then we say that they are *incident with* each other. We also say two faces are *adjacent* if they share at least one edge on the boundary. An edge with two incident vertices u and v is denoted by uv and a 3-face with three incident vertices u, v and w is denoted by uvw.

A vertex is a *neighbor* of another vertex if they are adjacent. The set of neighbors of a vertex v in graph G is denoted by $N_G(v)$. The degree of a vertex v is the number of its neighbors and it is denoted by $d(v)$. The degree of a face f is the length of the shortest boundary walk of f and it is denoted by $d(f)$. The girth of a graph is the length of a shortest cycle in the graph. A vertex with degree exactly k (resp., at least k or at most k) is called a *k-vertex* (resp., *k^+-vertex* or *k^--vertex*). Analogously, we can define *k-face*, *k^+-face* and *k^--face*. A vertex v is called a *k-neighbor* (resp., *k^+-neighbor* or *k^--neighbor*) of another vertex u if v is a neighbor of u and the degree of v is exactly k (resp., at least k or at most k).

The following concepts are important and are frequently used in our analysis. A neighbor u of v is a *pendant neighbor* of v if the edge uv is not incident with any 3-face. Let u be a pendant 3-neighbor of vertex v. A 3-face incident with u but not v is called a *pendant 3-face* of v. See Fig. 1(a) for an illustration. A 2-vertex is *bad* if it is incident with at least one 3-face and *good* otherwise. A 3-face is *bad* if it is incident with at least one 2-vertex and *good* otherwise. We further distinguish three kinds of bad 3-faces. A bad 3-face is called *terrible* if it is incident with a 5-vertex with three pendant 3^--neighbors. This 5-vertex in the terrible 3-face is also called an *abnormal* vertex of the 3-face. See Fig. 1(b) for an illustration of terrible

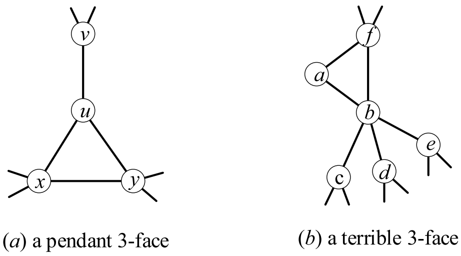

Fig. 1. (a): xyu is a pendant 3-face of v, where u is a 3-vertex; (b): abf is a terrible 3-face, where a is a 2-vertex, b is an abnormal vertex of the 3-face, and c, d and e are 3^--vertices.

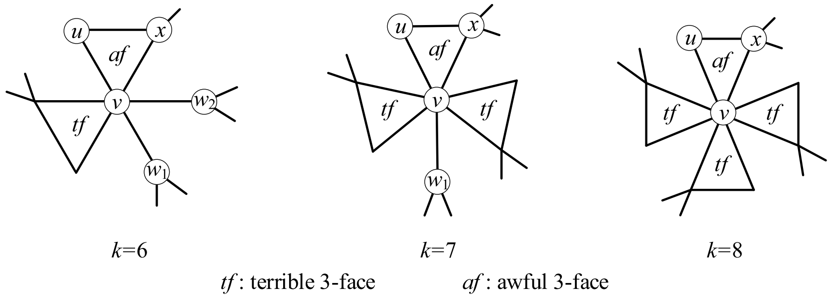

Fig. 2. Illustrations of awful 3-faces for $k = 6, 7, 8$, where vux is an awful 3-face, u is a 2-vertex, v is an abnormal k-vertex of vux, and w_1 and w_2 are 3^--vertices.

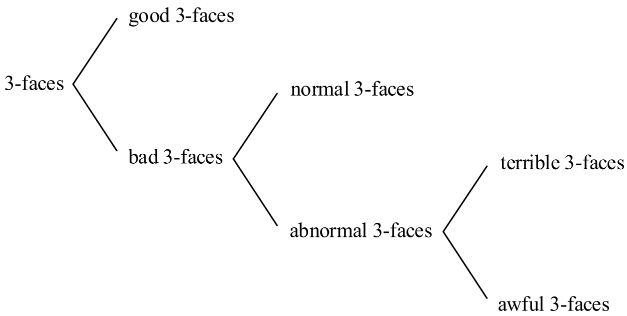

Fig. 3. Classification of 3-faces.

3-faces. A bad 3-face is called *awful* if it is incident with a k-vertex that is incident with $(k-5)$ terrible 3-faces and has $(8-k)$ pendant 3^--neighbors, where $k \in \{6, 7, 8\}$. This k-vertex in the awful 3-face is also called an *abnormal* vertex of the 3-face. See Fig. 2 for an illustration of awful 3-faces. A bad 3-face is called *abnormal* if it is terrible or awful and *normal* otherwise. For an abnormal bad 3-face, there are three vertices incident with it. One is a 2-vertex and one is an abnormal vertex. The last vertex is called the *third vertex* of the abnormal 3-face. Fig. 3 illustrates the classification of 3-faces.

3. Proof framework

Similar to the proofs in most previous papers in this research line, we prove our result by using the discharging method [19,8]. The main idea is as follows. We assume to the contrary that Theorem 1 is false, i.e., there is at least one planar graph without 4/5-cycles that is not (3, 3)-colorable. Then we look at a such kind of graph with the smallest size and use the discharging method to show some contradiction.

A planar graph without 4/5-cycles is called a *counterexample* if it is not (3,3)-colorable. A counterexample is *minimum* if the number of vertices is minimum and subject to that the number of edges is minimum. We assume that Theorem 1 is false and then the minimum counterexample exists. In the following part, we will let G be a minimum counterexample and analyze its properties.

We will initially assign a weight (charge) to each vertex and each face of the minimum counterexample G such that the sum of weights for all vertices and faces is negative. Then we design a set of rules to switch the weight among vertices and faces such that the sum of the total weight keeps unchanged. After the discharging operations, we can see that the weight of each vertex and each face is nonnegative, which is a contradiction to the fact that the total weight is negative. Different discharging methods may have different weight settings and different rules to switch weight (discharge). We should find a good setting to satisfy the requirement of our problem.

3.1. Discharging

In this paper, the initial charge of a vertex v and a face f is denoted by $\mu(v)$ and $\mu(f)$, respectively, which are defined as follows

$$\mu(v) = 2d(v) - 6 \quad \text{and} \quad \mu(f) = d(f) - 6. \tag{1}$$

By Euler's formula and the handshaking theorem, we have

$$\sum_{v\in V(G)} \mu(v) + \sum_{f\in F(G)} \mu(f) = \sum_{v\in V(G)} [2d(v) - 6] + \sum_{f\in F(G)} [d(f) - 6] = -12 < 0.$$

Next, we design a set of discharging rules and perform discharging processes by these rules. The final charge of each vertex v and each face f after the discharging processes will be denoted by $\mu^*(v)$ and $\mu^*(f)$, respectively. Note that the sum of charges of all vertices and faces keeps unchanged in our discharging processes. So it holds that

$$\sum_{v\in V(G)} \mu(v) + \sum_{f\in F(G)} \mu(f) = \sum_{v\in V(G)} \mu^*(v) + \sum_{f\in F(G)} \mu^*(f).$$

Our idea to design the discharging rules is to make the charge of each vertex and face nonnegative, i.e., $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$ for each vertex v and face f. If we can do this, then we have that $\sum_{v\in V(G)} \mu^*(v) + \sum_{f\in F(G)} \mu^*(f) \geq 0$, which is a contradiction to the fact that $\sum_{v\in V(G)} \mu(v) + \sum_{f\in F(G)} \mu(f) < 0$.

By the minimality of G, we know that in G, each vertex has a degree of at least 2 and each face has a degree of at least 3. According to (1), for the initial charge, only 2-vertices will have a negative charge and only 3-faces will have a negative charge (there are no 4/5-faces in G). So the purpose of our rules is to send charges to 2-vertices and 3-faces from other vertices and faces to make their charges nonnegative. We have seven discharging rules (R1)-(R7). Rule (R1) is to send charges from 5^+-vertices to 2-vertices, Rules (R2)-(R3) are to send charges from faces to 2-vertices, Rules (R4)-(R7) are to send charges from 4^+-vertices to 3-faces. The seven discharging rules are defined as follows:

(R1) Every 5^+-vertex sends 1 to each pendant 2-neighbor.
(R2) Every 7^+-face sends 1 to each incident bad 2-vertex.
(R3) Every 3-face sends 1 to each incident bad 2-vertex.
(R4) For every good 3-face, each incident 4^+-vertex sends 1 to it.
(R5) For every normal bad 3-face, each incident 5^+-vertex sends 2 to it.
(R6) For every abnormal bad 3-face, the abnormal vertex of it sends 1 to it and the third vertex of it sends 3 to it.
(R7) Every 5^+-vertex sends 1 to each pendant 3-face.

Please see Fig. 4 for illustrations of the discharging operations.

We have the following two important properties for the minimum counterexample G after the discharging processes.

Lemma 1. *It holds that $\mu^*(f) \geq 0$ for each face f in G.*

Lemma 2. *It holds that $\mu^*(v) \geq 0$ for each vertex v in G.*

We delay the proofs of the two lemmas to the next sections. By Lemmas 1 and 2, we know that the total charge of all vertices and faces in the minimum counterexample G is nonnegative. However, the initial total charge is negative and the total charge does not change after any discharging operations. This is a contradiction. So we know that the minimum counterexample G does not exist and then Theorem 1 holds.

Before proving Lemmas 1 and 2, we first show several structural properties of the minimum counterexample G that will be frequently used in our proofs.

4. Structural properties of minimum counterexample

In this section, we always assume that the graph is the minimum counterexample G. Clearly, G is a connected graph with the minimum degree of vertices at least 2 and the minimum degree of faces at least 3 by the minimality. Moreover, the graph G itself is not (3, 3)-colorable and any proper subgraph of G is (3, 3)-colorable. For a (3, 3)-coloring of any proper subgraph of G with the color set $\{1, 2\}$, a vertex colored $i \in \{1, 2\}$ is called *i-saturated* if it is adjacent to 3 neighbors colored with i. An i-saturated vertex is also simply called a *saturated vertex*.

The following three properties were observed in [17]. Note that if any of the three lemmas does not hold, then we would be able to find a 4/5-cycle in the graph.

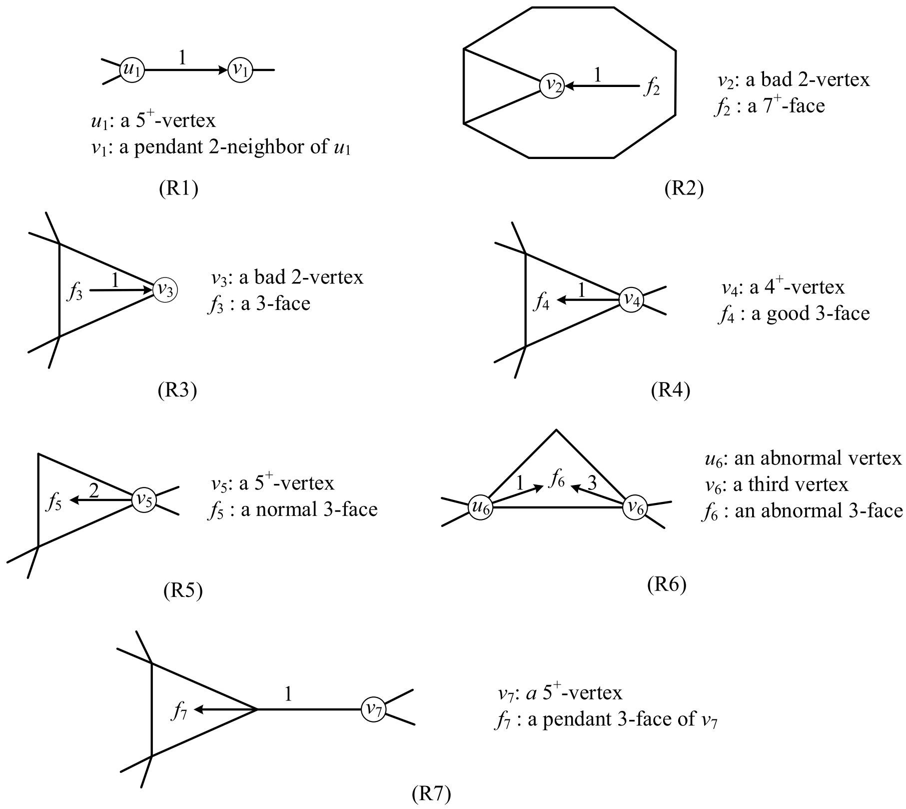

Fig. 4. Illustrations of the discharging operations, where an arrow indicates the direction of sending charges and the number beside the arrow is the charge.

Lemma 3. [17] *No two 3-faces share a common edge, and each k-vertex is incident with at most* $\lfloor \frac{k}{2} \rfloor$ *3-faces.*

Lemma 4. [17] *Let* v *be a* 2*-vertex incident with a* 3*-face in* G*. Then the other face incident with* v *is a* 7^+*-face.*

Lemma 5. [17] *Let* f *be a* k*-face with* $k \geq 7$ *in* G*. Then* f *has at most* $(k-6)$ *incident bad* 2*-vertices.*

Next, we prove several important lemmas.

Lemma 6. *No two* 4^-*-vertices are adjacent in* G*.*

Proof. Assume to the contrary that two 4^--vertices u and v are adjacent in G and we show a contradiction that G would be (3, 3)-colorable under this assumption. The proper subgraph $G - uv$ admits a (3, 3)-coloring c. Since c is not a $(3,3)$-coloring of G, we know that both of u and v are colored with the same color $i \in \{1,2\}$ and at least one of u and v is saturated in $G - uv$. Without loss of generality, we assume that u and v are colored with color 1 and u is 1-saturated in $G - uv$. Since u and v are 3^--vertices in $G - uv$, we know that u has exactly three neighbors in $G - uv$ that are all colored 1 in c. Thus, we can get another (3, 3)-coloring c' for $G - uv$ by recoloring u with 2 from c. Furthermore, c' is also a (3, 3)-coloring of G. It is a contradiction. □

Lemma 7. *Any* 7^-*-vertex is adjacent to at least one* 5^+*-vertex.*

Proof. Suppose otherwise that v is a k-vertex with all neighbors of degree of at most 4, where $k \leq 7$. By the minimality of G, we know that $G - v$ admits a (3, 3)-coloring c. If one neighbor u of v is saturated in c, then u is a 4-vertex in G, and all the other three neighbors of u are colored with the same color in c. We can recolor u with the other color so that u becomes un-saturated. So we can assume that all neighbors of v are not saturated in c. Since v has at most 7 neighbors, we know that there are at most 3 neighbors of v are colored with one color, say color 1. Thus, we can color v with color 1 to obtain a (3, 3)-coloring c' for G, a contradiction. □

Lemma 8. *Let* u *be a* 2*-vertex with two neighbors* v *and* w *in* G*. For any* (3, 3)*-coloring of* $G - u$*, one of* v *and* w *is* 1*-saturated and the other one is* 2*-saturated.*

This lemma holds because otherwise we could color u with 1 (if none of v and w is 1-saturated) or 2 (if none of v and w is 2-saturated) to get a $(3, 3)$-coloring of G.

Lemma 9. *Let uvw be a terrible 3-face in G, where u is a 2-vertex and v is an abnormal 5-vertex. For any (3, 3)-coloring c of $G_1 = G - \{vw, vu\}$, either v is unsaturated in c or we can recolor v with another color to make v unsaturated.*

Proof. We assume that v is saturated in c, say 1-saturated. Then the three neighbors of v in $N_G(v) \setminus \{u, w\}$ are colored with 1 in c. Then we can recolor v with 2 such that v is not saturated in G_1. □

Lemma 10. *Let uvw be an awful 3-face in G, where u is a 2-vertex and v is an abnormal k-vertex with $k \in \{6, 7, 8\}$. For any (3, 3)-coloring c of $G_1 = G - \{vw, vu\}$, either v is unsaturated in c or we can only recolor vertices in $N_G[v] \setminus \{u, w\}$ to make v unsaturated.*

Proof. We assume that vertex v is saturated in c, say 1-saturated, and try to recolor vertices in $N_G[v] \setminus \{u, w\}$ to make v unsaturated in G_1. Note that in G_1, vertex v is incident with exactly $(k - 5)$ terrible 3-faces and adjacent to exactly $(8 - k)$ pendant 3^--neighbors by the definition of awful 3-faces and abnormal vertices. We use $x_i y_i v$ $(i \in \{1, 2 \dots, k - 5\})$ to denote the $k - 5$ terrible 3-faces containing v, where x_i is the 2-vertex.

We look at graph $G_2 = G_1 - \{x_1 y_1, y_1 v\} \cup \{x_2 y_2, y_2 v\} \cup \cdots \cup \{x_{k-5} y_{k-5}, y_{k-5} v\}$, which is also colored with c. By Lemma 9, we know that we can get a (3,3)-coloring c' of G_2 such that all y_i are unsaturated by recoloring some (may none) of y_i from c.

If one of y_i, say y_{i^*} is colored with 2 in c', then we recolor all x_i $(i \in \{1, \dots, k - 5\})$ with 2 to get coloring c''. In c'', for each $i \in \{1, 2 \dots, k - 5\} \setminus \{i^*\}$, there is at most one vertex in $\{x_i, y_i\}$ which is colored with 1 and no vertex in $\{x_{i^*}, y_{i^*}\}$ is colored with 1. Thus, at most $(k - 5) - 1 + (8 - k) = 2$ vertices in $N_G(v) \setminus \{u, w\}$ are colored with 1 in c''. We can see that c'' is a (3,3)-coloring of G_1 such that v is unsaturated.

Next, we assume that all of y_i $(i \in \{1, \dots, k - 5\})$ are colored with 1 in c'. If at least one of the $(8 - k)$ pendant 3^--neighbors of v is colored with 2, then we recolor all x_i $(i \in \{1, \dots, k - 5\})$ with 2 to get coloring c''. In c'', at most $(k - 5) + (8 - k) - 1 = 2$ vertices in $N_G(v) \setminus \{u, w\}$ are colored with 1. We still get a (3,3)-coloring c'' of G_1 such that v is unsaturated. If all the $(8 - k)$ pendant 3^--neighbors of v are colored with 1, then we recolor all x_i $(i \in \{1, \dots, k - 5\})$ with 1 and recolor v with 2 to get coloring c''. In G_1, all neighbors of v are colored with 1 and v is colored with 2. Thus, c'' is a (3,3)-coloring of G_1 such that v is unsaturated. □

Lemma 9 and Lemma 10 directly imply the following lemma.

Lemma 11. *Let uvw be an abnormal 3-face in G, where u is a 2-vertex and v is an abnormal vertex of the 3-face. For any (3, 3)-coloring c of $G_1 = G - \{vw, vu\}$, either v is unsaturated in c or we can only recolor vertices in $N_G[v] \setminus \{u, w\}$ to make v unsaturated.*

Lemma 11 will be frequently used to prove other properties.

Lemma 12. *Each abnormal bad 3-face contains only one abnormal vertex.*

Proof. Suppose otherwise that an abnormal bad 3-face uvw contains two abnormal vertices, where u is a 2-vertex. Thus, both v and w are abnormal vertices of the 3-face. We look at the graph $G_1 = G - \{uv, uw, vw\}$ and let c be a (3,3)-coloring of G_1. We will apply Lemma 11 on v and w, separatively to make them unsaturated. Since there is no 4-cycle in G, we know that $N_G[v] \setminus \{u, w\}$ and $N_G[w] \setminus \{v, u\}$ are disjoint. By Lemma 11, we can recolor vertices in $N_G[v] \setminus \{u, w\}$ to make v unsaturated, and by Lemma 11 again, we can recolor vertices in $N_G[w] \setminus \{v, u\}$ to make w unsaturated. Now we assume that both of v and w are unsaturated. If v and w are colored with the same color, then we recolor u with the other color and thus this is a (3,3)-coloring of G. If v and w are colored with different colors, then we directly get a (3,3)-coloring of G no matter what color u is. In any case, we can get a (3,3)-coloring of G, a contradiction. □

Lemma 13. *The third vertex of each abnormal 3-face is a 6^+-vertex.*

Proof. Let uvw be an abnormal 3-face, where u is a 2-vertex, v is an abnormal vertex of the 3-face, and w is the third vertex. By Lemma 6, we know that w is 5^+-vertex. We only need to prove that w could not be a 5-vertex. Suppose otherwise that w is a 5-vertex. We look at the graph $G_1 = G - \{uv, uw, vw\}$ and let c be a (3,3)-coloring of G_1. By Lemma 11, we can recolor vertices in $N_G[v] \setminus \{u, w\}$ to make v unsaturated. If w is saturated in c, then all the three neighbors in $N_G[w] \setminus \{v, u\}$ are colored with the same color as that of w. For this case, we can recolor w with the other color to make w unsaturated. Thus, we can assume that both of v and w are unsaturated in c. If v and w are colored with the same color, then we recolor u with the other color and thus this is a (3,3)-coloring of G. If v and w are colored with different colors, then we directly get a (3,3)-coloring of G no matter what color u is. In any case, we can get a (3,3)-coloring of G, a contradiction. So w can not be a 5-vertex and then w is a 6^+-vertex. □

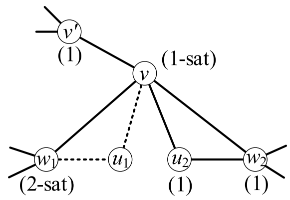

Fig. 5. A 5-vertex v is incident with two bad 3-faces u_1vw_1 and u_2vw_2, where u_1 and u_2 are bad 2-vertices. In the figure, the number in the bracket beside a vertex indicates the color of the vertex, and i-sat ($i \in \{1,2\}$) means the vertex is colored with i and it is saturated.

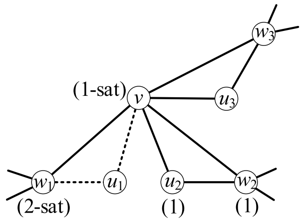

Fig. 6. A 6-vertex v is incident with three bad 3-faces u_1vw_1, u_2vw_2, and u_3vw_3, where u_1, u_2, and u_3 are bad 2-vertices. In the figure, the number in the bracket beside a vertex indicates the color of the vertex, and i-sat ($i \in \{1,2\}$) means the vertex is colored with i and it is saturated.

Lemma 14. *A 5-vertex is incident with at most one bad 3-face.*

Proof. Suppose otherwise that a 5-vertex v is incident with two bad 3-faces u_1vw_1 and u_2vw_2, where u_1 and u_2 are bad 2-vertices. We let v' be the last neighbor of v not in the two bad 3-faces. See Fig. 5 for an illustration. By the minimality, we know that $G - u_1$ has a (3, 3)-coloring c. By Lemma 8, we know that one of v and w_1 is 1-saturated and the other is 2-saturated in c. Without loss of generality, we assume that v is 1-saturated and w_1 is 2-saturated. Then we can see that u_2, w_2 and v' are colored with 1 in c. Then we can recolor u_2 with 2. Now v is not 1-saturated. We can color u_1 with 1 to obtain a $(3,3)$-coloring for G, a contradiction. □

Lemma 15. *A 6-vertex is incident with at most two bad 3-faces.*

Proof. Suppose otherwise that a 6-vertex v is incident with three bad 3-faces u_1vw_1, u_2vw_2, and u_3vw_3, where u_1, u_2, and u_3 are bad 2-vertices. See Fig. 6 for an illustration. By the minimality, we know that $G - u_1$ has a (3, 3)-coloring c. By Lemma 8, we know that one of v and w_1 is 1-saturated and the other is 2-saturated in c. Without loss of generality, we assume that v is 1-saturated and w_1 is 2-saturated. So three vertices in $\{u_2, w_2, u_3, w_3\}$ are colored with 1 and then either u_2 and w_2 or u_3 and w_3 are colored with 1 in c. Without loss of generality, we assume that u_2 and w_2 are colored with 1. Now we can recolor u_2 with 2 and v becomes not 1-saturated. We can further color u_1 with 1 to obtain a $(3,3)$-coloring for G, a contradiction. □

Lemma 16. *Let v be a k-vertex that is not an abnormal vertex of any 3-face, where $6 \le k \le 8$. Then v is incident with at most $(k-5)$ abnormal 3-faces.*

Proof. Suppose otherwise that v is incident with at least $(k-4)$ abnormal 3-faces, where $6 \le k \le 8$. Thus, v is incident with at least two abnormal 3-faces since $k \ge 6$. Denote the $(k-4)$ abnormal 3-faces incident with v by $u_1vw_1, \ldots, u_{k-4}vw_{k-4}$, where $u_1, \ldots, u_{k-4}$ are 2-vertices and $w_1, \ldots, w_{k-4}$ are abnormal vertices. The other $8-k$ neighbors of v are called *lateral vertices* and denoted by $v_1, \ldots, v_{8-k}$. For any abnormal vertex w_i $(1 \le i \le k-4)$ and any lateral neighbor v_j $(1 \le j \le 8-k)$, they have no common neighbor, otherwise there would be a 4-cycle in G. For any two different abnormal vertices w_{i_1} and w_{i_2} $(1 \le i_1, i_2 \le k-4)$, the intersection of $N_G[w_{i_1}] \setminus \{v, u_{i_1}\}$ and $N_G[w_{i_2}] \setminus \{v, u_{i_2}\}$ is empty, otherwise there would be a 4-cycle in G. These properties will allow us to apply Lemma 11 on each w_i independently without affecting other neighbors of v.

By Lemma 8, we can assume with loss of generality that there is a (3, 3)-coloring c of $G_0 = G - \{u_1v, u_1w_1\}$ such that v is 1-saturated and w_1 is 2-saturated in c. By Lemma 11, we can recolor w_1 and some of its neighbors to get a (3, 3)-coloring c_0 of $G_1 = G - \{u_1v, u_1w_1, vw_1\}$ such that w_1 is unsaturated. Note that w_1 should not be colored with 2 in c_0, otherwise we can recolor u_1 with 2 to get a (3,3)-coloring of G, a contradiction. So we assume that in G_1, there is a (3, 3)-coloring c_0 such that v is 1-saturated and w_1 is 1-unsaturated.

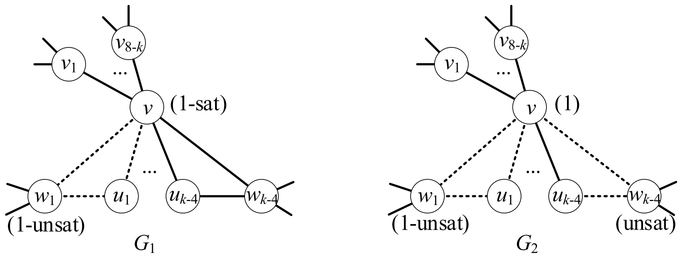

Fig. 7. An illustration of G_1 and G_2 in Lemma 16. In the figure, the number in the bracket beside a vertex indicates the color of the vertex, i-sat ($i \in \{1, 2\}$) means the vertex is colored with i and it is saturated, and i-unsat ($i \in \{1, 2\}$) means the vertex is colored with i and it is not saturated.

We look at the graph $G_2 = G_1 - \{w_2u_2, w_2v\} \cup \cdots \cup \{w_{k-4}u_{k-4}, w_{k-4}v\}$ with the (3, 3)-coloring c_0. By Lemma 11, for each w_i with $i \in \{2, \ldots, k-4\}$, we can recolor w_i and some of its neighbors to make it unsaturated (note that in G_2, w_i and w_j ($i \neq j$) have no common neighbor). Thus, we can assume that in G_2, there is a (3, 3)-coloring c_1 such that v is colored with 1, w_1 is 1-unsaturated, and all w_i ($i \in \{2, \ldots, k-4\}$) are unsaturated. See Fig. 7 for an illustration of G_1 and G_2.

We further show that none of w_i is colored with 2 in c_1. Assume to the contrary that w_{i^*} ($i^* \in \{2, \ldots, k-4\}$) is 2-unsaturated. Then we recolor all u_i ($i \in \{2, \ldots, k-4\}$) with 2. We look at graph G_1. In G_1, vertex v has at most $(8-k) + (k-5) - 1 = 2$ neighbors colored with 1, which are the $(8-k)$ lateral neighbors and w_i with $i \in \{2, \ldots, k-4\} \setminus \{i^*\}$. Thus, we can recolor u_1 with 2 to get a (3,3)-coloring of G, a contradiction. We can also assume that all the $(8-k)$ lateral neighbors of v are colored with 1. Because if one of the lateral neighbors is colored with 2, then in G_1, the number of neighbors of v colored with 1 is at most $(8-k-1) + (k-5) = 2$ (after recoloring all u_i ($i \in \{2, \ldots, k-4\}$) with 2). We can recolor u_1 with 2 to get a (3,3)-coloring of G, a contradiction. So next we assume that in G_2, there is a (3, 3)-coloring c_1 such that v is colored with 1, all w_i ($i \in \{1, 2, \ldots, k-4\}$) are 1-unsaturated, and all lateral neighbors v_j ($j \in \{1, \ldots, 8-k\}$) are colored with 1.

For coloring c_1, we recolor all u_i ($i \in \{1, 2, \ldots, k-4\}$) with 1 and recolor v with 2 to get a coloring c_2. We can see that c_2 is a (3, 3)-coloring of G, a contradiction. In any case, we can always get a contradiction that there is a (3, 3)-coloring of G. Thus, the lemma holds. □

Lemma 17. *No k-vertex is incident with $(k-5)$ abnormal bad 3-faces and adjacent to $(10-k)$ pendant 3^--neighbors, where $6 \le k \le 10$.*

Proof. Suppose otherwise that a k-vertex v ($6 \le k \le 10$) is incident with $(k-5)$ abnormal 3-faces $\{u_1vw_1, \ldots, u_{k-5}vw_{k-5}\}$ and $(10-k)$ pendant 3^--neighbors $\{v_1, \ldots, v_{10-k}\}$, where $u_1, \ldots, u_{k-5}$ are 2-vertices and $w_1, \ldots, w_{k-5}$ are abnormal vertices. For any abnormal vertex w_i ($1 \le i \le k-5$) and any pendant neighbor v_j ($1 \le j \le 10-k$), they have no common neighbor, otherwise there would be a 4-cycle in G. For any two different abnormal vertices w_{i_1} and w_{i_2} ($1 \le i_1, i_2 \le k-5$), the intersection of $N_G[w_{i_1}] \setminus \{v, u_{i_1}\}$ and $N_G[w_{i_2}] \setminus \{v, u_{i_2}\}$ is empty, otherwise there would be a 4-cycle in G. These properties will allow us to apply Lemma 11 on each w_i independently without affecting other neighbors of v.

In $G_0 = G - \{u_1v, u_1w_1\}$, there is a (3, 3)-coloring c such that v is 1-saturated and w_1 is 2-saturated by Lemma 8. We can further assume that w_1 is unsaturated in $G_1 = G - \{u_1v, u_1w_1, vw_1\}$ by Lemma 11. If w_1 is 2-unsaturated, then we could recolor u_1 with 2 to get a (3,3)-coloring of G, a contradiction. So we know that w_1 is 1-unsaturated. Thus, in G_1, there is a (3, 3)-coloring c_1 such that v is 1-saturated and w_1 is 1-unsaturated. We distinguish two cases.

Case 1: $k = 10$, i.e., v is a 10-vertex in G. Now in G_1, vertex v is a 8-vertex contained in four abnormal 3-faces $\{u_2vw_2, \ldots, u_5vw_5\}$. In G_1 with coloring c_1, vertex v has only three neighbors colored with 1 since it is 1-saturated. Thus, we can assume without loss of generality that both of u_5 and w_5 are colored with 2 in c_1. We look at the graph $G_2 = G_1 - \{w_2u_2, w_2v\} \cup \{w_3u_3, w_3v\} \cup \{w_4u_4, w_4v\}$ with the (3, 3)-coloring c_1. By Lemma 11, for each w_i with $i \in \{2, 3, 4\}$, we can recolor w_i and some of its neighbors to make it unsaturated. Thus, we can assume that in G_2, there is a (3, 3)-coloring c_2 such that v is colored with 1, w_1 is 1-unsaturated, u_5 and w_5 are colored with 2, and all w_i ($i \in \{2, 3, 4\}$) are unsaturated.

If one vertex in $\{w_2, w_3, w_4\}$ is colored with 2, then we recolor all vertices in $\{u_2, u_3, u_4\}$ with 2. Thus, there are at most two vertices in $N_G(v) \setminus \{u_1, w_1\}$ are colored with 1. We can get a (3,3)-coloring of G by recoloring u_1 with 2. Otherwise all vertices in $\{w_2, w_3, w_4\}$ are colored with 1. For this case, we recolor all vertices in $\{u_2, u_3, u_4\}$ with 1. Now there are two vertices in $N_G(v) \setminus \{u_1, w_1\}$ are colored with 2, which are u_5 and w_5. We can get a (3, 3)-coloring of G by recoloring v with 2, u_5 with 1 (to make sure that the number of neighbors of w_5 colored with 2 will not increase), and u_1 with 1, a contradiction. See Fig. 8 for an illustration of Case 1.

Case 2: $6 \le k \le 9$. Let $G_2 = G - \{u_1v, u_1w_1, vw_1\}$ for $k = 6$ and $G_2 = G - \{u_1v, u_1w_1, vw_1\} \cup \{w_2u_2, w_2v\} \cup \cdots \cup \{w_{k-5}u_{k-5}, w_{k-5}v\}$ for $k \ge 7$. By Lemma 11, we can further assume that in G_2, each w_i ($i \in \{2, \ldots, k-5\}$ and $k \ge 7$) is unsaturated. We also recolor u_i ($i \in \{2, \ldots, k-5\}$) with the color different from the color of w_i. We note that $v_1, \ldots, v_{10-k}$

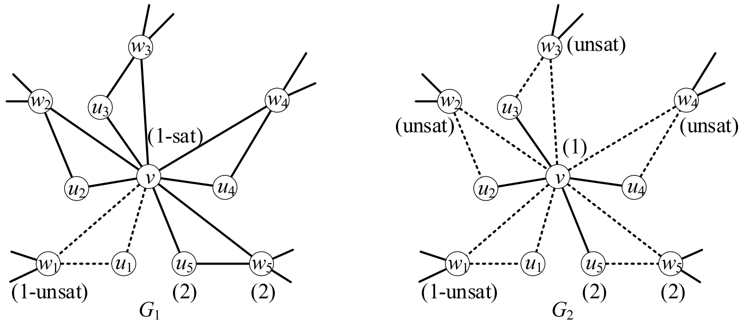

Fig. 8. An illustration of G_1 and G_2 in Case 1 of Lemma 17. In the figure, the number in the bracket beside a vertex indicates the color of the vertex, i-sat ($i \in \{1,2\}$) means the vertex is colored with i and it is saturated, and i-unsat ($i \in \{1,2\}$) means the vertex is colored with i and it is not saturated.

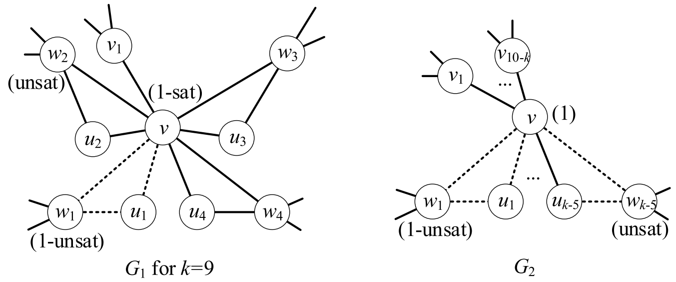

Fig. 9. An illustration of G_1 and G_2 in Case 2 of Lemma 17. In the figure, the number in the bracket beside a vertex indicates the color of the vertex, i-sat ($i \in \{1,2\}$) means the vertex is colored with i and it is saturated, and i-unsat ($i \in \{1,2\}$) means the vertex is colored with i and it is not saturated.

are pendant 3^--neighbors and they have at most two neighbors besides v. Recovering of v just makes them from saturated to unsaturated. If at most three vertices in $N_G(v) \setminus \{u_1, w_1\}$ are colored with 2, then we could get a (3,3)-coloring of G by recoloring v with 2 and u_1 with 1. If at most two vertices in $N_G(v) \setminus \{u_1, w_1\}$ are colored with 1, then we could get a (3,3)-coloring of G by recoloring v with 1 and u_1 with 2. So $|N_G(v) \setminus \{u_1, w_1\}| \geq 4+3=7$ and then $k=9$.

So v is a 9-vertex in G. In G_1, vertex v is a 7-vertex with four neighbors colored with 2 and three neighbors colored with 1. If one vertex $u_i \in \{u_2, u_3, u_4\}$ is colored with 1, then we simply recolor u_i with 2. Now only two vertices in $N_G(v) \setminus \{u_1, w_1\}$ are colored with 1, and then we can get a (3,3)-coloring of G by recoloring u_1 with 2, a contradiction. So all vertices in $\{u_2, u_3, u_4\}$ are colored with 2. For this case, we can also get a (3,3)-coloring of G by recoloring u_2, u_3, u_4 with 1 and recoloring v with 2, a contradiction. See Fig. 9 for an illustration of Case 2. □

5. Proofs of Lemma 1 and Lemma 2

Now we are ready to prove Lemmas 1 and 2. First, we consider Lemma 1. We show that any face f in G has a nonnegative charge after the recharging processes.

Proof. Recall that G has only 3-faces and 6^+-faces since there are no 4/5-cycles.

Case 1. Face f is a 3-face. Initially we have that $\mu(f) = 3-6 = -3$. Discharging rules (R3)-(R7) involve 3-faces. Only (R3) takes some charges from the 3-face and all other rules are going to increase the charge of the 3-face. There are three cases for the 3-face f.

Case 1.1. Face f is an abnormal bad 3-face. The face f will send 1 to the incident 2-vertex by (R3), get 1 from the abnormal vertex of it by (R6), and get 3 from the third vertex of it by (R6). Thus, $\mu^*(f) = -3-1+1+3 = 0$.

Case 1.2. Face f is a normal bad 3-face. There is a 2-vertex incident with it. By Lemma 6, we know that the other two vertices incident with the face are 5^+-vertices. The face f will send 1 to the incident 2-vertex by (R3), and get 2 from each of the other two incident 5^+-vertices by (R5). Then $\mu^*(f) = -3-1+2\times 2 = 0$.

Case 1.3. Face f is a good 3-face. By Lemma 6, we know that f is incident with at most one 3-vertex. If f is not incident with any 3-vertex, then each of the three incident 4^+-vertices will send 1 to f by (R4) and $\mu^*(f) = -3+3\times 1 = 0$. If f is incident with one 3-vertex u, then all neighbors of u are 5^+-vertices by Lemma 6. Let v be the third neighbor of u not incident with f. Then f is a pendant 3-face of v and f will get 1 from v by (R7). The other two vertices incident with f are 5^+-vertices and each of them will send 1 to f by (R4). Thus, $\mu^*(f) = -3+1+2\times 1 = 0$.

Case 2. Face f is a 6-face. Initially we have that $\mu(f) = 6 - 6 = 0$. No recharging rule involves a 6-face. So its charge keeps unchanged.

Case 3. Face f is a k-face with $k \geq 7$. Initially we have that $\mu(f) = k - 6$. Only (R2) involves a k-face. By Lemma 5, we know that f has at most $k - 6$ incident bad 2-vertices, each of which may take at most 1 from f by (R2). Thus, $\mu^*(f) \geq k - 6 - (k - 6) = 0$.

For any face f, it holds that $\mu^*(f) \geq 0$, and then Lemma 1 holds. □

Second, we prove Lemma 2. We show that any vertex v in G has a nonnegative charge after the recharging processes.

Proof. Recall that any vertex in G has a degree of at least 2. We will prove the lemma by considering different cases according to the degree of the vertex v. Note that in G, a k-vertex v is incident with most $\lfloor \frac{k}{2} \rfloor$ 3-faces according to Lemma 3.

Case 1. Vertex v is a 2-vertex. Initially we have that $\mu(v) = 4 - 6 = -2$. First, we assume that v is a bad 2-vertex. By Lemma 4, we know that v is incident with a 3-face and a 7^+-face. Then v can get 1 from each of the two incident faces by (R2) and (R3). Thus, $\mu^*(v) = -2 + 1 + 1 = 0$. Second, we assume that v is a good 2-vertex. By Lemma 6, we know that v has two 5^+-neighbors, each of which will give 1 to v by (R1). Thus, $\mu^*(v) = -2 + 1 + 1 = 0$.

Case 2. Vertex v is a 3-vertex. Initially we have that $\mu(v) = 6 - 6 = 0$. We can see that none of our recharging rules involves a 3-vertex. So its charge always keeps unchanged.

Case 3. Vertex v is a 4-vertex. Initially we have that $\mu(v) = 8 - 6 = 2$. Note that the third vertex of an abnormal bad 3-face is a 6^+-vertex by Lemma 13 and then (R6) does not involve a 4-vertex. Only rule (R4) involves a 4-vertex. Vertex v is incident with at most two 3-faces by Lemma 3, each of which may take 1 from v by (R4). So $\mu^*(v) \geq 2 - 2 \times 1 = 0$.

Case 4. Vertex v is a 5-vertex. Initially we have that $\mu(v) = 10 - 6 = 4$. There are five discharging rules (R1), (R4), (R5), (R6) and (R7) that involve 5-vertices. We know that v is incident with at most two 3-faces by Lemma 3 again. We consider the following three subcases.

Case 4.1. Vertex v is not incident with any 3-face. By Lemma 7, we know that v is adjacent to at most four 3^--neighbors, each of which may take at most 1 from v by (R1) or (R7). Thus, $\mu^*(v) \geq 4 - 4 \times 1 = 0$.

Case 4.2. Vertex v is incident with exactly one 3-face f. If f is not a terrible 3-face or a good 3-face, then v is adjacent to at most two pendant 3^--neighbors, each of which may take at most 1 from v by (R1) or (R7). The 3-face f may also take at most 2 from v by (R5). So $\mu^*(v) \geq 4 - 2 \times 1 - 2 = 0$. If f is a terrible 3-face, then v is an abnormal vertex adjacent to three pendant 3^--neighbors, each of which may take at most 1 from v by (R1) or (R7). The 3-face f may take at most 1 from v by (R6). So, $\mu^*(v) \geq 4 - 3 \times 1 - 1 = 0$. If f is a good 3-face, then v is adjacent to at most three pendant 3^--neighbors, each of which may take at most 1 from v by (R1) or (R7). The 3-face f takes 1 from v by (R4). So $\mu^*(v) \geq 4 - 3 \times 1 - 1 = 0$.

Case 4.3. Vertex v is incident with two 3-faces. By Lemma 14, we know that at least one 3-face is good. According to the definition of terrible and awful 3-faces and Lemma 13, we know that if a 5-vertex is incident with an abnormal bad 3-face, then the 3-face can only be a terrible 3-face and the 5-vertex is the abnormal vertex of the face. However, these do not hold for the vertex v. So none of the two 3-faces is an abnormal bad 3-face. We have that $\mu^*(v) \geq 4 - 2 - 1 - 1 = 0$, because each incident normal bad 3-face may take at most 2 from v by (R5), each incident good 3-face may take at most 1 from v by (R4), and there is at most one pendant 3^--neighbor that may take at most 1 from v by (R1) or (R7).

Case 5. Vertex v is a k-vertex with $k \in \{6, 7, 8\}$. Initially we have that $\mu(v) = 2k - 6$. Vertex v is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces. If v is the abnormal vertex of an abnormal 3-face, then $\mu^*(v) \geq (2k - 6) - 3(k - 5) - 1 - (k - 2(k - 4)) = 0$ since v will send 3 to an incident abnormal 3-face if v is not the abnormal vertex of the face and 1 to an incident abnormal 3-face if v is the abnormal vertex of the face. If v is not an abnormal vertex of an abnormal 3-face, we know that v is incident with at most $(k - 5)$ abnormal 3-faces by Lemma 16.

If v is incident with a good 3-face, then $\mu^*(v) \geq (2k - 6) - 3(k - 5) - 1 - (k - 2(k - 4)) = 0$. Next, we assume that v is not an abnormal vertex of any abnormal 3-face and v is not incident with any good 3-face. We consider the following four cases.

Case 5.1. Vertex v is incident with at most $(k - 6)$ abnormal 3-faces. Then $\mu^*(v) \geq (2k - 6) - 3(k - 6) - (k - 2(k - 6)) = 0$.

Case 5.2. Vertex v is incident with exactly $(k - 5)$ abnormal 3-faces and v is not incident with any normal 3-face. By Lemma 17, we know that v is adjacent to at most $(9 - k)$ pendant 3^--neighbors. Then $\mu^*(v) \geq (2k - 6) - 3(k - 5) - (9 - k) = 0$.

Case 5.3. Vertex v is incident with exactly $(k - 5)$ abnormal 3-faces and v is incident with exactly one normal 3-face. Since v is not an abnormal vertex of an awful 3-face, we know that v is adjacent with at most $(7 - k)$ pendant 3^--neighbors. Then $\mu^*(v) \geq (2k - 6) - 3(k - 5) - 2 - (7 - k) = 0$.

Case 5.4. Vertex v is incident with exactly $(k - 5)$ abnormal 3-faces and v is incident with at least two normal 3-faces. Since v is incident with at least $(k - 3)$ 3-faces, by Lemma 3 we know that k can only be 6 and then v is a 6-vertex. However, by Lemma 15, we know that no 6-vertex is incident with three bad 3-faces. So this case is impossible.

Case 6. Vertex v is a 9-vertex. Initially we have that $\mu(v) = 18 - 6 = 12$. We know that v is incident with at most four 3-faces. If v is incident with at most three 3-faces, then $\mu^*(v) \geq 12 - 3 \times 3 - 3 \times 1 = 0$, since each incident 3-face will take at most 3 from v by (R6) and each pendent neighbor or pendent 3-face will take at most 1 from v by (R1) or (R7). If v is incident with four 3-faces and at least one is not abnormal, then $\mu^*(v) \geq 12 - 3 \times 3 - 2 - 1 = 0$, since each incident abnormal 3-face will take at most 3 from v by (R6), each other incident 3-face will take at most 2 from v by (R4) or (R5), and each pendent neighbor or pendent 3-face will take at most 1 from v by (R1) or (R7). If v is incident with four abnormal 3-faces, then v is not adjacent to any pendant 3^--neighbor by Lemma 17. We have that $\mu^*(v) \geq 12 - 3 \times 4 = 0$.

Case 7. Vertex v is a 10-vertex. Initially we have that $\mu(v) = 20 - 6 = 14$. We know that v is incident with at most five 3-faces. If v is incident with five 3-faces, then at least one is not an abnormal 3-face by Lemma 17. We have $\mu^*(v) \geq 14 - 3 \times 4 - 2 = 0$ because each incident abnormal 3-face will take at most 3 from v by (R6), each incident good or normal 3-face will take at most 2 from v by (R4)-(R5). If v is incident with at most four 3-faces, then $\mu^*(v) \geq 14 - 4 \times 3 - 2 = 0$.

Case 8. Vertex v is a k-vertex with $k \geq 11$. Initially we have that $\mu(v) = 2k - 6$. We know that v is incident with at most $\lfloor \frac{k}{2} \rfloor$ 3-faces, each of which may take at most 3 from v by (R6), and each pendent neighbor or pendent 3-face will take at most 1 from v by (R1) or (R7). If k is odd, then $\mu^*(v) \geq (2k - 6) - \frac{k-1}{2} \times 3 - 1 = \frac{k-11}{2} \geq 0$. If k is even, then $\mu^*(v) \geq (2k - 6) - \frac{k}{2} \times 3 = \frac{k-12}{2} \geq 0$.

For any vertex v, it holds that $\mu^*(v) \geq 0$, and then Lemma 2 holds. □

6. Concluding remarks

Grötzsch Theorem and Steinberg's conjecture are good issues to consider the (improper) colorable of planar graphs. Now Steinberg's conjecture has been disproved. For three colors, whether every planar graph without 4/5-cycles is (1,0,0)-colorable is the only unknown case now. However, for two colors, the gap seems still big. We know that every planar graph without 4/5-cycles is $(2, 6)$-colorable and $(3, 3)$-colorable but some planar graphs without 4/5-cycles are not $(1,k)$-colorable for each $k \geq 0$. It will be interesting to further reduce the gap. Another interesting question is that almost all previous proofs in this research line can only show the colorability of the graph and we still do not know how to color the graph and even do not know whether it is computationally hard or not. It is also worthy to study algorithms and computational complexity for coloring the graphs.

Declaration of competing interest

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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